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On a Simple Wave Approximation of a Set of Linear Dispersive Wave Equations

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SUMMARY

The validity of an approximation α_0 of one of the solutions α of a set of two linear coupled dispersive wave equations has been discussed. α_0 is the solution of a linear Korteweg–de Vries equation and satisfies the same initial condition as α. It is shown that for square integrable solutions having a spectral range not exceeding $[-\Delta, \Delta]$ the approximation is useful if $\Delta^5\mu^2 t \ll 1$ in the sense that $\|\alpha-\alpha_0\|(t) \ll \|\alpha\|(t)$ (L_2-norm). μ is a measure for the dispersion. The approximation fails in that sense as $t\to\infty$. Some remarks to a similar nonlinear problem are made.

1. Introduction

In two papers, [1] and [2], L. J. F. Broer and the present author have considered a set of two linear coupled dissipative wave equations. We were interested especially in the range of validity of an approximation of this set applying to a certain class of initial value problems. The approximation "leads" to a linear Burgers' equation. In this paper, a similar approximation for a set of linear dispersive wave equations will be treated. This set is given by

$$\alpha_t+\alpha_x = -\mu(\alpha+\beta)_{xxx}\,, \tag{1}$$

$$\beta_t-\beta_x = \mu(\alpha+\beta)_{xxx}\,, \tag{2}$$

where μ is a positive constant and the subscript t (or x) denotes partial differentiation with respect to $t(x)$.

An example of such a set is furnished by an intermediate representation of the equations describing the longitudinal motion of an infinite chain of identical masses and springs. By an intermediate representation we mean a representation "between" the exact continuum representation and the lowest continuum limit (cf. [3]). It is given by

$$u_{tt} = c^2 u_{xx} + \frac{a^2c^2}{12}u_{xxxx}\,, \tag{3}$$

where a is the lattice constant and c is the propagation speed of waves in the lowest continuum limit, i.e. $a\to 0$.

For reasons of uniqueness of the exact continuum representation (see [3]), it is necessary that u is square integrable and has a spectral range not exceeding $[-\pi a^{-1}, \pi a^{-1}]$. Then, as is shown in [3] too, stability is also assured.

Substituting $\alpha=-u_t+cu_x$, $\beta=u_t+cu_x$ and putting $c=1$ (which can be done without any loss of generality), from (3) we find (1) and (2) with $\mu=a^2/24$.

The approximation we shall study applies to the class of initial value problems

$$\alpha(x,0)=f(x)\,, \tag{4}$$

$$\beta(x,0)=0\,. \tag{5}$$

When $\mu=0$, it is seen that (2) is satisfied identically. Then (1) becomes a first order equation

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in α, which is easily solved. The resultant solution is a simple wave solution for the hyperbolic set obtained by putting $\mu=0$.

Now, the approximation, which as in [1] and [2] will be called the simple wave approximation henceforth, is based on the assumption that, when the initial conditions (4) and (5) are prescribed for the equations (1) and (2) with μ small but not zero, β will be negligible, at any rate for some finite interval of time. In this way one obtains from (1)

$$\alpha_t+\alpha_x=-\mu\alpha_{xxx}. \tag{6}$$

This method of approximation has been used by Zabusky [4] in his theory of wave propagation in a nonlinear one-dimensional lattice. He obtains the Korteweg–de Vries (KdV) equation. (6) is the linearised form of that equation.

Now, the problem is that β will grow slowly from zero and therefore it is not at all obvious that α satisfies (6) for longer intervals of time. In section 4 the range of validity of the simple wave approximation and an expansion of α and β will be considered. Some mathematical notations and the representations of the solutions α, β and α_0 needed there, will be given in sections 2 and 3. α_0 is the solution of (6) subject to (4). The situation as $t\to\infty$ is discussed in section 5 and the last section is devoted to some remarks concerning a similar problem for nonlinear equations.

2. Mathematical Notations

R is the interval $(-\infty, \infty)$ of the real numbers. Consider scalar-valued complex functions $u(x)$ defined on R.

$L_2(R)$ is a Hilbert-space containing all square integrable functions on R with inner product $(\ ,\ )$ and norm $\|\ \ \|$ defined by

$$(u, v)=\int_{-\infty}^{\infty} u^*(x)\,v^*(x)\,dx\ ;\quad \|u\|=(u, u)^{\frac{1}{2}},$$

where u^* is the complex conjugate of u.

The space $L_2^{\Delta}(R)$ is a Hilbert-space containing all functions $u\in L_2(R)$ of which the Fourier transform $\bar{u}(k)$ defined by

$$\bar{u}(k)=\int_{-\infty}^{\infty} u(x)\exp(-ikx)\,dx$$

vanishes identically outside a finite interval $[-\Delta, \Delta]$ $(\Delta\in R)$.

The inner product $(\ ,\ )_{R,\Delta}$ and norm $\|\ \ \|_{R,\Delta}$ are defined by

$$(u, v)_{R,\Delta}=\int_{-\infty}^{\infty} u^*(x)\,v(x)\,dx\ ;\ \|u\|_{R,\Delta}=(u, u)_{R,\Delta}^{\frac{1}{2}}.$$

Where not stated otherwise all integrations are in the sense of Lebesque.

3. $L_2^{\Delta}(R)$ Solutions

Let $f\in L_2^{\Delta}(R)$. Assume, for reasons of uniqueness of the exact continuum representation (see section 1 and [3]), $\Delta\leq\frac{1}{2}\pi(6\mu)^{-\frac{1}{2}}$ henceforth. As may be verified now easily, α and β are given by

$$\alpha=\frac{1}{2\pi}\left[\int_{-\Delta}^{\Delta}1+\int_{-\Delta}^{\Delta}2\right]\frac{(k+\omega)^2}{4\omega k}\bar{f}(k)\exp(ikx-i\omega t)\,dk, \tag{1}$$

$$\beta=\frac{1}{2\pi}\left[\int_{-\Delta}^{\Delta}1+\int_{-\Delta}^{\Delta}2\right]\frac{k^2-\omega^2}{4\omega k}\bar{f}(k)\exp(ikx-i\omega t)\,dk, \tag{2}$$

where

$$\omega(k)=k(1-2\mu k^2)^{\frac{1}{2}}$$

and the numbers 1 and 2 through the integration symbol mean integration in the first—respectively second sheet of the complex k—plane. The first sheet is defined by

$$\lim_{|k|\to\infty} \frac{\omega(k)}{k^2} = -i\sqrt{(2\mu)} \qquad (0 \leqq \arg k \leqq \pi)$$

and the second by

$$\lim_{|k|\to\infty} \frac{\omega(k)}{k^2} = i\sqrt{(2\mu)} \qquad (0 \leqq \arg k \leqq \pi)\,.$$

Finally

$$\alpha_0 = \frac{1}{2\pi}\int_{-\Delta}^{\Delta} \bar{f}(k)\exp(ikx - ikt + i\mu k^3 t)\,dk \tag{3}$$

4. The Validity of the Simple Wave Approximation

4.1. *Periodic solutions*

Consider the periodic initial condition

$$f(x) = \exp(ik_1 x)$$

where $k_1 \in R$ and $|k_1| \leqq \frac{1}{2}\pi(6\mu)^{-\frac{1}{2}}$.

The solutions α, β and α_0 may formally be found by substituting $\bar{f}(k) = 2\pi\delta(k-k_1)$ in (3.1), (3.2) and (3.3). Therefore

$$\alpha = \frac{(1+c)^2}{4c}\exp(ikx - ikct) - \frac{(1-c)^2}{4c}\exp(ikx + ikct)\,, \tag{1}$$

$$\beta = \frac{1-c^2}{4c}\exp(ikx - ikct) + \frac{c^2-1}{4c}\exp(ikx + ikct)\,, \tag{2}$$

$$\alpha_0 = \exp(ikx - ikt + i\mu k^3 t)\,,$$

where

$$c(k) = |(1-2\mu k^2)^{\frac{1}{2}}|$$

and the subscript 1 has been omitted again.

The formulae (1) and (2) clearly demonstrate the development of left- and right moving waves, whereas α_0 consists of a right travelling wave only.

Substitution of $\sin(kct) = 1/2i[\exp(ikct) - \exp(-ikct)]$ in (1) and (2) leads to

$$\alpha = \exp(ikx - ikct) - \tfrac{1}{2}i\frac{(1-c)^2}{c}\sin(kct)\exp(ikx)\,,$$

$$\beta = \tfrac{1}{2}i\frac{c^2-1}{c}\sin(kct)\exp(ikx)\,,$$

showing that α may also be seen as a superposition of a right moving- and a standing, β as a pure standing wave. Expanding $(1-2\mu k^2)^{\frac{1}{2}}$ around $k=0$ gives

$$\alpha = [1 + \tfrac{1}{2}i\mu^2 k^5 t + \ldots]\exp[ikx - ikt + i\mu k^3 t] + \\ + [-\tfrac{1}{2}i\mu^2 k^4 \sin(kt) + \ldots]\exp(ikx)\,.$$

From this equation we infer that, if $\mu^2|k|^5 t \ll 1$,

$$|\alpha - \alpha_0| \ll |\alpha_0| = 1\,, \tag{3}$$

so we may speak of a good simple wave approximation in that sense.

If $\mu k^2 \ll 1$, the left travelling part of the α-mode is still small compared with the remaining part as $t \to \infty$. However, the difference between the phases of α's right moving part and α_0 may be large and therefore (3) fails to hold. If one is not interested in the relative phases mentioned, as is often the case in dealing with periodic waves, one may still speak of a useful approximation. A similar problem will arise in dealing with $L_2^4(R)$ solutions.

4.2. *Expansion in a series of $L_2^4(R)$ solutions*

To get some more insight in the character of solutions of equations like (1.1) and (1.2), one often uses expansions in a series. We shall construct such an expansion of α, taking as the first term in the series α_0. The method we shall use is entirely similar to that used in [1], therefore all details will be stripped. Let $f \in L_2^4(R)$. Introduce the operators

$$M = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu \frac{\partial^3}{\partial x^3},$$

$$N = \frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \mu \frac{\partial^3}{\partial x^3},$$

Then, (1.1) and (1.2) become

$$M\alpha = -\mu\beta_{xxx},$$

$$N\beta = \mu\alpha_{xxx},$$

so

$$MN\alpha = -\mu \frac{\partial^3}{\partial x^3} N\beta = -\mu^2 \frac{\partial^6 \alpha}{\partial x^6},$$

which implies that α and β satisfy

$$\left.\begin{aligned} L\alpha &= -\mu^2 \frac{\partial^6 \alpha}{\partial x^6}, \\ L\beta &= -\mu^2 \frac{\partial^6 \beta}{\partial x^6} \end{aligned}\right. \tag{4}$$

respectively.

L is given by

$$L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - 2\mu \frac{\partial^4}{\partial x^4} - \mu^2 \frac{\partial^6}{\partial x^6}.$$

The initial data become

$$\left.\begin{aligned} \alpha(x,0) &= f(x), \\ \alpha_t(x,0) &= -\frac{df(x)}{dx} - \mu \frac{d^3 f(x)}{dx^3}, \end{aligned}\right\} \tag{5}$$

$$\beta(x,0) = 0,$$

$$\beta_t(x,0) = \mu \frac{d^3 f(x)}{dx^3}.$$

The solution of (4) and (5) satisfies

$$\alpha = \alpha_0 + A\alpha, \tag{6}$$

where

$$A\alpha = \frac{\mu^2}{2\pi}\int_0^t d\tau \int_{-\Delta}^{\Delta} dk\, k^5 \bar{\alpha}(k,\tau)\frac{\sin[(k-\mu k^3)(t-\tau)]}{1-\mu k^2}\exp(ikx)\,.$$

In a similar way, we find that β satisfies

$$\beta = \mu\beta_1 + A\beta\,, \tag{7}$$

where

$$\beta_1 = \frac{-1}{2\pi}\int_{-\Delta}^{\Delta} ik^3 \bar{f}(k)\frac{\sin[(k-\mu k^3)t]}{k-\mu k^3}\exp(ikx)\,dk\,.$$

(6) and (7) may be solved by means of iteration:

$$\begin{aligned}
&\alpha^{(0)} = \alpha_0\,,\\
&\alpha^{(2n)} = \mu^{-2}A\alpha^{(2n-2)}\,,\\
&\beta^{(1)} = \beta_1\,,\\
&\beta^{(2n-1)} = \mu^{-2}A\beta^{(2n-3)} \qquad (n=1,2,\ldots)\,.
\end{aligned} \tag{8}$$

Now, starting from (8) it may be shown, similar as was done in [1] and choosing the function $q^2(k,t)$ used there equal to

$$\exp\left[-\frac{\mu^4 k^{10}}{(1-\mu k^2)^2}e^t - t\right],$$

that, for all finite $t \geqq 0$,

$$\sum_{n=0}^{N}\mu^{2n}\alpha^{(2n)}$$

converges to α,

$$\sum_{n=0}^{N}\mu^{2n+1}\beta^{(2n+1)}$$

converges to β as $N\to\infty$ in the sense of the $L_2^\Delta(R)$ norm. Furthermore

$$\left\|\alpha - \sum_{n=0}^{N}\mu^{2n}\alpha^{(2n)}\right\|_{R,\Delta} \leqq \left\{\sum_{n=N+1}^{\infty}\left(\frac{\Delta^5\mu^2 t}{1-\mu\Delta^2}\right)^n\frac{1}{n!}\right\}\|\alpha\|_{R,\Delta}\,. \tag{9}$$

4.3. *The simple wave approximation*

We shall call $\alpha_0 \in L_2^\Delta(R)$ a good simple wave approximation to $\alpha\in L_2^\Delta(R)$ in the interval of time $[t_1, t_2]$ if and only if for every $t\in[t_1,t_2]$

$$\|\alpha-\alpha_0\|_{R,\Delta} \ll \|\alpha\|_{R,\Delta}\,. \tag{10}$$

According to (9), (10) is satisfied for $t\in[0,T]$ where $\mu^2\Delta^5 T \ll 1$. This result is entirely similar to that found in case of periodic solutions discussed in section 4.2. Now, we shall show that (10) is certainly not satisfied for all $L_2^\Delta(R)$ solutions as $t\to\infty$. This is in constrast with the result we found in [1] for the simple wave approximation of the dissipative set of equations.

By using Parseval's theorem we find from (3.1), (3.2) and (3.3)

$$\|\alpha-\alpha_0\|^2_{R,\Delta} = 2\|\alpha_0\|^2_{R,\Delta} + \|\beta\|^2_{R,\Delta} +$$

$$+\frac{1}{2\pi}\int_{-\Delta}^{\Delta}\{-(k+\omega)^2\cos[(\omega-\omega_0)t] + (k-\omega)^2\cos[(\omega+\omega_0)t]\}\frac{|\bar{f}|^2}{2k\omega}\,dk \tag{11}$$

where

$$\omega_0 = k - \mu k^3\,,$$
$$\|\alpha_0\|_{R,\Delta} = \|f\|_{R,\Delta}\,,$$
$$\|\beta\|^2_{R,\Delta} = \frac{1}{2\pi}\int_{-\Delta}^{\Delta} \frac{(k^2-\omega^2)^2}{4k^2\omega^2}\sin^2(\omega t)|\bar{f}|^2\,dk\,.$$

In these formulae ω will be chosen in the first sheet of the complex k-plane. We may also write

$$\|\beta\|^2_{R,\Delta} = \frac{1}{2\pi}\int_{-\Delta}^{\Delta} \frac{(k^2-\omega^2)^2}{8k^2\omega^2}|\bar{f}|^2\,dk - \frac{1}{2\pi}\int_{-\Delta}^{\Delta} \frac{(k^2-\omega^2)^2}{8k^2\omega^2}\cos(2\omega t)|\bar{f}|^2\,dk\,. \tag{12}$$

ω and $\omega+\omega_0$ may have two-, $\omega-\omega_0$ three points of stationary phase for $k\in[-\Delta, \Delta]$. They are located symmetrically with respect to $k=0$.

Assuming that $\bar{f}(k)$ is of bounded variation in $[-\Delta, \Delta]$, we find by applying the method of stationary phase to (11) and (12) (Lauwerier [5]) and using the lemma of Riemann–Lebesque

$$\lim_{t\to\infty} \|\alpha-\alpha_0\|^2_{R,\Delta} = 2\|f\|^2_{R,\Delta} + \frac{1}{2\pi}\int_{-\Delta}^{\Delta} \frac{(k^2-\omega^2)^2}{8k^2\omega^2}|\bar{f}|^2\,dk\,.$$

It is thus proved that (10) does not hold for all $L_2^\Delta(R)$ solutions as $t\to\infty$. The result is due to the oscillatory character of the solution for large t and will become more clear in the next section.

5. Asymptotic Behaviour as $t\to\infty$

Let $f\in L_2^\Delta(R)$ and $\bar{f}(k)$ analytic in $(-\Delta, \Delta)$. Write

$$\alpha = \alpha_1 + \alpha_2\,,$$

where

$$\alpha_j = \frac{1}{2\pi}\int_{-\Delta}^{\Delta} j\,\frac{(k+\omega)^2}{4\omega k}\bar{f}(k)\exp[ih(k,\xi)t]\,dk \qquad (j=1,2)\,,$$
$$h(k,\xi) = k\xi - \omega(k)\,,$$
$$\xi = xt^{-1}\,.$$

We shall study the asymptotic behaviour of α_1 as $t\to\infty$ by means of the method of stationary phase. The function α_2 may be treated in a similar way.

Let, from now on until stated otherwise, all functions defined in the k-plane, be defined in the first sheet of that plane. Let ε and δ be positive, but arbitrary small, numbers. The points of stationary phase of $h(k,\xi)$ are solutions of

$$\xi = v(k)\,,$$

where the group velocity $v(k)=d\omega/dk$ is given by

$$v(k) = (1-4\mu k^2)(1-2\mu k^2)^{-\frac{1}{2}}\,.$$

If $v(\Delta)\leqq\xi\leqq 1$, two such points exist (say) $\bar{k}(\xi)$ and $-\bar{k}(\xi)(\bar{k}\geqq 0)$. Outside that range of ξ-values there is none. So, if $-\infty<\xi\leqq v(\Delta)-\delta$ or $1+\varepsilon\leqq\xi<\infty$ we find by means of partial integration:

$$\alpha_1 = \frac{[\Delta+\omega(\Delta)]^2}{8\pi i\Delta\omega(\Delta)[\xi-v(\Delta)]t}\{\bar{f}(\Delta)e^{i[\Delta x-\omega(\Delta)t]}+\bar{f}(-\Delta)e^{-i[\Delta x-\omega(\Delta)t]}\}+O(t^{-2})(t\to\infty)\,. \tag{1}$$

According to Copson [6], the method of stationary phase yields, if $v(\Delta)+\delta \leqq \xi \leqq 1-\varepsilon$,

$$\alpha_1 = [-2\pi v'(\bar{k})t]^{-\frac{1}{2}} \frac{[\bar{k}+\bar{\omega}]^2}{4\bar{\omega}\bar{k}} \{\bar{f}(\bar{k}) e^{i[\bar{k}x-\bar{\omega}t+\pi/4]} +$$

$$+\bar{f}(-\bar{k}) e^{-i[\bar{k}x-\bar{\omega}t+\pi/4]}\} + O(t^{-1})(t\to\infty)\,, \qquad (2)$$

where $\bar{\omega}=\omega(\bar{k})$ and $v'(k)$ is the first derivative of $v(k)$.

When $\xi\to v(\Delta)$, we have $|\bar{k}|\to\Delta$. Therefore, the domain $v(k)-\delta<\xi<v(k)+\delta$ has been omitted from the range of ξ-values. Another method is necessary in that case. However, as these values of ξ are relatively unimportant, we shall not proceed in that direction.

As $\xi\to 1$, $v'(k)\to 0$. So, only if $\bar{f}(0)=0$ is satisfied, (1) may be used for $v(\Delta)+\delta\leqq\xi\leqq 1$. Then, thanks to the analyticity of $\bar{f}$ in a vicinity of $k=0$, $\bar{f}(k)=O(k)$ as $k\to 0$.

Now, let $\bar{f}(0)\neq 0$.

Theorem 1.

Define $\eta=\xi-1$. Let $|\eta t|\leqq 1$. As $t\to\infty$,

$$\alpha_1 = \bar{f}(0)(3\mu t)^{-\frac{1}{3}} Ai[(3\mu t)^{-\frac{1}{3}}\eta t] + O(t^{-\frac{2}{3}})\,, \qquad (3)$$

where the Airy-function $Ai(x)$ is defined by

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}y^3+xy\right)dy\,.$$

Proof.

$\bar{f}(k)$ is regular for $|k|<\Delta$. Split the interval of integration in three parts: $[-\Delta, -p)$, $[-p, p]$ and $(p, \Delta]$ where $0<p<\Delta$. By means of partial integration we see that the contributions of the first- and third interval are $O(t^{-1})$ as $t\to\infty$.

Introduce a new variable u by means of

$$h(k,\xi) = \eta u + \mu u^3\,.$$

If $\eta=0$ this equation has one, otherwise three solutions

$$k = \sum_{n=1}^{\infty} b_n u^n\,,$$

all regular in a vicinity of $u=0$.

Choose that solution for which b_1 is real, so $b_1=1$.

It follows that

$$\frac{(k+\omega)^2}{4\omega k}\bar{f}(k)\frac{dk}{du} = \sum_{n=0}^{\infty} c_n u^n\,,$$

where $c_0=\bar{f}(0)$. Write

$$\frac{(k+\omega)^2}{4\omega k}\bar{f}(k)\frac{dk}{du} = \bar{f}(0)+u\Psi(u)\,,$$

so, as $t\to\infty$,

$$\alpha_1 = \frac{1}{2\pi}\int_{-q}^{q} \{\bar{f}(0)+u\Psi(u)\}\exp[iu\eta t+i\mu u^3 t]\,du + O(t^{-1})\,, \qquad (4)$$

where

$$q = u(p)\,.$$

We may write (4) in the form

$$\alpha_1 = \frac{\bar{f}(0)}{2\pi}\int_{-\infty}^{\infty} \exp[iu\eta t + i\mu u^3 t]\,du + I + O(t^{-1}) \qquad (t\to\infty),$$

where

$$I = \frac{1}{2\pi}\int_{-q}^{q} u\Psi(u)\exp[iu\eta t + i\mu u^3 t]\,du - \frac{\bar{f}(0)}{2\pi}\left[\int_{-\infty}^{-q} + \int_{q}^{\infty}\right]\exp[iu\eta t + i\mu u^3 t]\,du\,.$$

In the appendix it is proved that $I = O(t^{-\frac{2}{3}})$ as $t\to\infty$. This proves the theorem.

The theorem gives information about the asymptotic behaviour in the shaded region of fig. 1.

Still, we don't have information about the regions A and B. Most important of course is A.

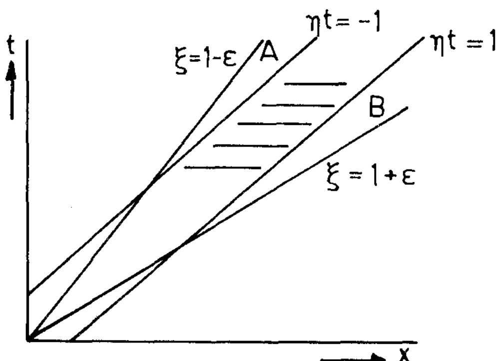

Figure 1.

Theorem 2.

As $t\to\infty$, the range of validity of (2) and (3) can be extended to A.

Proof.

We shall demonstrate that formal expansions of (2) and (3) fit together in A. Let $\eta t < 0$. Expand with respect to large η, in particular (t fixed) $-\eta t(3\mu t)^{-\frac{1}{3}} \gg 1$. From Abramowitz and Stegun [7] we obtain

$$Ai(-z) = \pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \sin\left(\tfrac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right) + O(z^{-\frac{7}{4}}),$$

as $z\to\infty$. Using this result we find from (3) as $t\to\infty$, $-\eta t^{\frac{2}{3}}\to\infty$:

$$\alpha_1 \sim \bar{f}(0)(-3\pi^2\eta\mu t^2)^{-\frac{1}{4}} \sin\left[-\tfrac{2}{3}\eta\left(\frac{-\eta}{3\mu}\right)^{\frac{1}{2}} t + \frac{\pi}{4}\right]. \tag{5}$$

Expand all quantities in (2) for small η. We find as $\eta\to 0$

$$\bar{k} \sim \left(\frac{-\eta}{3\mu}\right)^{\frac{1}{2}}, \quad v'(\bar{k}) \sim -2(-3\mu\eta)^{\frac{1}{2}}, \quad ht \sim \tfrac{2}{3}\eta t\left(\frac{-\eta}{3\mu}\right)^{\frac{1}{2}}$$

and so, for $\eta\to 0$, $t\to\infty$, (5) may be deduced again. This proves the theorem.

Speaking in terms of singular perturbation theory one may call (2) the outer- and (3) the inner solution. Then, η is interpreted as the large outer- and small inner variable. In fact we have made an outer expansion of the inner solution and the reverse and demonstrated they fit together (Kaplun [8]).

If one is able to construct asymptotic expansions of α_1 for $|\xi - 1| \leqq \varepsilon$, $|\eta t| \leqq 1$ and for $1+\varepsilon \leqq$

$\xi<\infty$, it must surely be possible to fit them together in B. However, this leads to severe mathematical difficulties lying beyond the scope of this paper.

For α_2 we also find (1) and (2), but now all functions are defined in the second sheet of the k-plane*. As $(\omega+k)^2/\omega k=O(k^4)$ as $|k|\to 0$ in that sheet, (1) holds for $v(\Delta)+\delta\leqq\xi<\infty$ and $-\infty<\xi\leqq -1$, (2) for $-1\leqq\xi\leqq v(\Delta)-\delta$. Now, we may sketch the wave phenomenon.

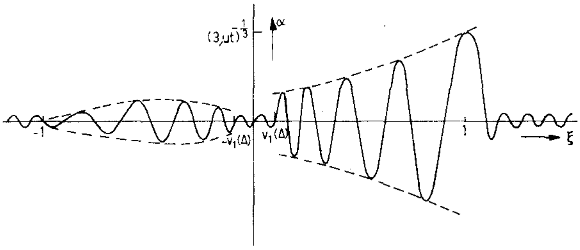

Figure 2. $\bar{f}(k)=1$ $(|k|\leqq\Delta)$, $\Delta<\frac{1}{2}\sqrt{\mu}$ $v_1(\Delta)=(1-4\mu\Delta^2)/|1-2\mu\Delta^2)^{\frac{1}{2}}|$.

The front of the right travelling part of the α-wave is formed by the "Airy wave". Such a wave may described by a carrier wave which is amplitude modulated. However, the carrier wave has a wavelength infinitely longer than that of the modulation. This kind of waves also constitutes the so called tidal waves. The steepness of the front of the Airy wave and the "wavelength" directly behind the front increase as the cube root of t.

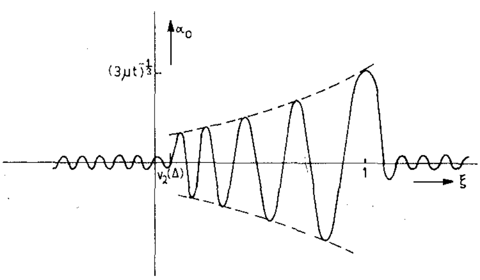

Figure 3. $\bar{f}(k)=1(|k|\leqq\Delta)$, $\Delta<(3\mu)^{-\frac{1}{2}}$, $v_2(\Delta)=1-3\mu\Delta^2$.

The asymptotic behaviour of α_0 can be described in a similar way. The result is sketched in fig. 3. It turns out that if $\mu\Delta^2\ll 1$, α and α_0 resemble each other for $x\geqq 0$ in the sense that local amplitude and wavelength only slightly differ. Then, the left moving part of the α-mode is small (in the maximum norm) compared with the right moving part. Therefore, in that sense, one may call α_0 a good approximation of α as $t\to\infty$. This result resembles that of section 4.2 very much. This section also explains the remark made at the end of section 4.3.

6. Some Critical Remarks on a Nonlinear Case

In this section we want to devote some attention to a simple wave approximation of a set of nonlinear equations. Let that set be given by

$$\alpha_t+[1+\varepsilon(\alpha+\beta)]\alpha_x=-\mu(\alpha+\beta)_{xxx}\,, \tag{1}$$

$$\beta_t-[1+\varepsilon(\alpha+\beta)]\beta_x=\mu(\alpha+\beta)_{xxx}\,, \tag{2}$$

and the initial conditions by (1.4) and (1.5).

Then, by a similar reasoning as used in section 1, we may argue that for some finite interval of time the behaviour of the α-mode approximately is described by the solution α_0 of

* Of course, k is now the negative solution of $\xi=v(k)$.

$$\alpha_{0t}+\alpha_{0x}+\varepsilon\alpha_0\alpha_{0x}=-\mu\alpha_{0xxx}\,, \tag{3}$$

$$\alpha_0(x,0)=f(x)\,. \tag{4}$$

We suppose again that $f\in L_2^4(R)$.

Equations (1) and (2) are used by Zabusky [4] as an intermediate representation for the equations describing the behaviour of a nonlinear one-dimensional lattice. Then, by the further approximation indicated, he finds (3), which is the KdV equation. This equation has been studied by him and several other authors [9, 10, 11] extensively. It is also used as a long wavelength approximation in various fields of physics such as the theory of cold plasmas and shallow water theory (see [11, 12, 13]).

Here we want to make some remarks on equations (1) and (2) and the corresponding simple wave approximation. First, it is not clear at all whether the equations (1) and (2) subject to (1.4) and (1.5) have stable solutions. The solution α_0 of (3) subject to (4) is stable. By stability we mean that a positive definite norm for the solution exists such that, uniformly with respect to time, it is bounded in terms of the corresponding norms of the initial conditions. (1.1) and (1.2) gave rise to the same problem. However, in that case, stability is assured due to the boundedness of the spectral range of the solutions (see [3]). The solutions of (1) and (2) subject to (1.4) and (1.5) have an unbounded spectral range for each $t>0$.

This unboundedness also leads to the remark that the simple wave approximation probably will break down even faster than in the linear case. However, one should be very careful stating that conjecture as, especially for large times t, the solution of the KdV equation (3) subject to (4) is of an entirely different character than that of the linearised version (1.6). The non-linear solution consists of solitons which are steady progressive solutions of (3).

They result from a balance between the dispersive- and nonlinear effects. Nevertheless, when we are *far* before breakdown time, that is the time at which the solution of (3) where $\mu=0$ starts developing a shock wave, the solution of (3) and its linear version probably will look very much alike (cf. [4]). In that case, the conjecture made above, presumably is useful.

Appendix

We shall prove that I, defined in section 4, equals $O(t^{-\frac{2}{3}})$ as $t\to\infty$. Denoting

$$P(u;\,\eta t,\mu t)=\exp[i\eta tu+i\mu tu^3]$$

and writing

$$u\Psi(u)=c_1u+u^2\chi(u)\,, \tag{1}$$

we have

$$I=\frac{c_1}{2\pi}\int_{-\infty}^{\infty}uP\,du-\frac{1}{2\pi}\left[\int_{-\infty}^{-q}+\int_q^{\infty}\right][\bar f(0)+c_1u]P\,du+\frac{1}{2\pi}\int_{-q}^{q}u^2\chi(u)P\,du\,.$$

Choose $q\leqq 1$. Now, using partial integration twice

$$\left|\int_q^{\infty}[\bar f(0)+c_1u]P\,du\right|\leqq\left|\int_q^{\infty}\frac{ic_1\eta t}{u}\frac{P}{3i\mu t}du\right|+$$

$$+\frac{|\bar f(0)|+|c_1|q}{3\mu tq^2}+\left|\int_q^{\infty}\left(\frac{-2\bar f(0)}{u^3}-\frac{c_1}{u^2}+\frac{i\bar f(0)\eta t}{u^2}\right)\frac{P}{3i\mu t}du\right|\leqq$$

$$\leqq\frac{|c_1||\eta t|}{3\mu t}\left\{\frac{1}{3\mu tq^3}+\left|\int_q^{\infty}\left(\frac{i\eta t}{u^3}-\frac{3}{u^4}\right)\frac{P}{3i\mu t}du\right|\right\}+$$

$$+\frac{[3|\bar f(0)|+2|c_1|]}{3q^2\mu t}\leqq\frac{[3|\bar f(0)|+2|c_1|]}{3q^2\mu t}+\frac{|c_1|}{3q^4\mu^2t^2}\,.$$

In a similar way, we may estimate

$$\int_{-\infty}^{-q} [\bar{f}(0)+c_1 u] P\,du \,.$$

By partial integration we also find

$$\left|\int_{-q}^{q} u^2 \chi(u) P\,du\right| \leqq \max_{u=\pm q} \frac{|\chi(q)|}{3\mu t} + \left|\int_{-q}^{q} \left[i\eta t\,\chi(u) + \frac{d\chi}{du}\right] \frac{P}{3i\mu t}\,du\right| \leqq$$

$$\leqq \text{constant.}\ (3\mu t)^{-1} \,.$$

At this place, the reason for the further splitting (1) becomes clear. Upon partial integration of

$$\int_{-q}^{q} u\Psi(u) P\,du \,,$$

we would have introduced a pole in the new integrand.

Finally

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} uP(u;\ \eta t,\ \mu t)\,du = -i(3\mu t)^{-\frac{2}{3}} \left[\frac{dAi(\xi)}{d\xi}\right]_{\xi=(3\mu t)^{-1/3}\eta t} ,$$

from which the statement made at the beginning of this appendix immediately follows.

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